LOCAL SIDON SETS AND UNIFORM CONVERGENCE OF FOURIER SERIES

BY

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ABSTRACT

For compact groups several necessary and sufficient conditions for a set to be local Sidon are given; these conditions are expressed in functional-analytic terms. An immediate corollary is the existence of a large class of compact groups G, including all the connected non-Abelian Lie groups, which support functions not in $A(G)$ but having Fourier series that are both uniformly and absolutely convergent. This has been shown for the special case of $SU(2)$ by Mayer. Also several necessary and sufficient conditions for a set to be local $\Lambda(p)$ are given.

1. Introduction

Throughout we suppose that G is a Hausdorff compact group and that Γ denotes a maximal set of inequivalent continuous representations of G which are unitary and irreducible. The symbols C and L^p will denote the spaces of continuous functions and p-integrable functions on G respectively and their respective norms will be denoted by $\|\cdot\|$ and $\|\cdot\|_p$. If $f \in L^1$, its Fourier series will be written as

$$f \sim \sum_{\gamma \in \Gamma} d(\gamma)\,\mathrm{tr}\,[\hat{f}(\gamma)\gamma(\,\cdot\,)]$$

where $d(\gamma)$ is the degree of the representation γ, tr is the usual trace function, and $\hat{f}(\gamma) = \int_G f(x)\gamma(x^{-1})dx$, dx being the normalised Haar measure. The space $A(G)$, or simply A, is defined as the set of f in C such that

$$\|f\|_A \equiv \Sigma_\gamma\, d(\gamma)\mathrm{tr}[|\hat{f}(\gamma)|] < \infty.$$

We are now able to follow Rider [6] and give the definitions of local Sidon and local $\Lambda(p)$ sets.

DEFINITIONS. A set $E \subseteq \Gamma$ is said to be *local Sidon* if there exists a number $B > 0$ such that

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$$\|f\|_A \leqq B\|f\| \tag{1.1}$$

for all functions f of the form

$$f = d(\gamma)\mathrm{tr}[\hat{f}(\gamma)\gamma], \ \gamma \in E. \tag{1.2}$$

A set $E \subseteq \Gamma$ is said to be *local* $\Lambda(p)$, $p > 0$, if there exists a number B such that, for some r satisfying $0 < r < p$,

$$\|f\|_p \leqq B\|f\|_r \tag{1.3}$$

for all functions satisfying (1.2).

PROPOSITION 1.1.

(i) *A set* $E \subseteq \Gamma$ *is local* $\Lambda(p)$ *if and only if to every* r *satisfying* $0 < r < p$ *there exists* B *such that* (1.3) *is satisfied for all functions of the form* (1.2).

(ii) *A local Sidon set is local* $\Lambda(p)$ *for all* $p > 0$.

(iii) *The union of two local Sidon sets (respectively, local* $\Lambda(p)$ *sets) is a local Sidon set (respectively, local* $\Lambda(p)$ *set).*

PROOF. The proof of Part (i) is accomplished by using Hölder's inequality (see [3, p. 421]); that of Part (ii) can be deduced from a powerful result due to Figà-Talamanca and Rider (see [2] and also [3, Th. (37.10) and its proof]); while that of Part (iii) is trivial.

The necessary and sufficient conditions for a set to be local Sidon or local $\Lambda(p)$ will, in the main, be expressed in terms of the spaces CC and CL^p. We say that $f \in CC$ or that f has a *convergent-in-norm* Fourier series (respectively, $f \in CL^p$ or that f has a *convergent-in-p-norm* Fourier series) if $f \in L^1$ and

$$\|f\|_{CC} \equiv \Sigma\, d(\gamma)\,\|\mathrm{tr}[\hat{f}(\gamma)\gamma]\| < \infty$$

(respectively,

$$\|f\|_{CL^p} \equiv \Sigma\, d(\gamma)\,\|\mathrm{tr}[\hat{f}(\gamma)\gamma]\|_p < \infty).$$

Clearly $\|\cdot\|_{CC}$ and $\|\cdot\|_{CL^p}$ are norms on CC and CL^p respectively under which they become Banach spaces.

It is easily seen that every convergent-in-norm Fourier series is both uniformly and absolutely convergent. When B is abelian, (unconditional) uniform convergence, absolute convergence, and convergence-in-norm are each equivalent to membership of A; also every subset of Γ is local Sidon. On the other hand, when $G = SU(2)$, [4, Th. 4.1] shows the existence of a function in CC which is not a member of A, while [5] shows that every infinite subset of Γ is not a a local $\Lambda(p)$ set for any $p > 0$ and hence not a local Sidon set. These results are linked by the prototype proposition that *$E \subseteq \Gamma$ is a local Sidon set if and only if every E-spectral function in CC is a member of A.*

One of the main reasons for the study of local lacunary sets is that it serves as a bridge to the study of (global) lacunary sets. For example, see [1], [5] and [6].

2. Necessary and sufficient conditions

Necessary and sufficient conditions for a set to be Sidon or $\Lambda(p)$ are easily given in terms of Banach spaces (or their norms) of E-spectral functions or measures and consequently their theory is amenable to the techniques of functional analysis. (See [3, Sect. 37], for example.) In this section we show that conditions of this type are also possible for local lacunary sets. Whenever E is a subset of Γ and X is a set of functions or measures, we denote the set of E-spectral members of X by X_E.

THEOREM 2.1. *The following conditions on $E \subseteq \Gamma$ are equivalent.*

(i) *E is local Sidon.*

(ii) *CC_E is contained in A (and hence $CC_E = A_E$).*

(iii) *There exists $B = B(E)$ such that $\|f\|_A \leqq B\|f\|_{CC}$ for all $f \in CC_E$.*

PROOF. Clearly (iii) implies (ii) so that their equivalence follows from the closed-graph theorem. Suppose that E is local Sidon and that $f \in CC_E$. Then, using (1.1) for each of the functions $f_\gamma = d(\gamma)\mathrm{tr}[\hat{f}(\gamma)\gamma]$, $\gamma \in E$, we have

$$\|f\|_A = \Sigma_\gamma \|f_\gamma\|_A \leqq B\ \Sigma_\gamma \|f_\gamma\| = B\|f\|_{CC}.$$

On the other hand, suppose that (iii) is satisfied. Then, in particular, $\|f\|_A \leqq B\|f\|_{CC} = B\|f\|$ for all f satisfying (1.2). Thus (i) and (ii) are also equivalent.

Before proceeding to the second type of condition equivalent to local sidonicity we introduce several spaces. Let $\mathfrak{E}^\infty = \mathfrak{E}^\infty(\Gamma)$ denote the set of functions ϕ on Γ such that

(i) $\phi(\gamma)$ is an automorphism of the Hilbert space of dimension $d(\gamma)$, and

(ii) $$\|\phi\|_\infty \equiv \sup\{\|\phi(\gamma)\|_{(\infty)} : \gamma \in \Gamma\} < \infty,$$

where $\|\phi(\gamma)\|_{(\infty)}$ is the norm of $\phi(\gamma)$ as an operator. Denote A', the space of continuous linear functionals on A, by P. Then P is called the space of pseudo-measures on B and can be put in one-to-one correspondence with $\mathfrak{E}^\infty$; the element $\sigma \in P$ corresponds with $\phi \in \mathfrak{E}^\infty$ provided

(2.1) $$\langle f,\sigma\rangle = \sum d(\gamma)\operatorname{tr}[\bar{f}^{\wedge}(\gamma) * \phi(\gamma)]$$

for all $f \in A$ where $*$ denotes the adjoint operation. In this case we will denote $\phi(\gamma)$ by $\hat{\sigma}(\gamma)$. Given $\sigma_1, \sigma_2 \in P$, their convolution is defined as the unique element $\sigma_1 * \sigma_2$ of P which has the Fourier transform $\hat{\sigma}_2\hat{\sigma}_1(\in \mathfrak{E}^\infty)$.

Each continuous linear functional on CC becomes a member of P when its domain is restricted to A; thus CC' may be thought of as a space lying between P and the space of measures M on G. We define the Fourier transform $\hat{\mu}$ of each $\mu \in CC'$ via (2.1). Finally each $f \in L^1$ will be thought of as a member of P via the functional $g \mapsto \int_G fg$ on A. Then

(2.2) $$A \subseteq CC \subseteq C \subseteq M \subseteq CC' \subseteq P,$$

and each of the imbedding maps has norm 1. Furthermore it is easily checked that the notions of convolution and Fourier transform defined above are true extensions from the cases of functions and measures. (The convolution $f * g$ of L^1 functions f and g is defined to be the function $x \mapsto \int_G f(y)g(y^{-1}x)dx$.)

Given $f \in L^1$, we define the left translation $\tau_a f$ and the right translation $\rho_a f$ of f by $a \in G$ as the functions $x \mapsto f(ax)$ and $x \mapsto f(xa)$ respectively. Also f^* and $f^\vee$ will denote the functions $x \mapsto \overline{f(x^{-1})}$ and $x \mapsto f(x^{-1})$ respectively.

LEMMA 2.2.

(i) *For each $f \in CC$ and $a \in G$, the CC norms of $\tau_a f$, $\rho_a f$, $f^\vee$, f^* and f are all equal.*

(ii) *Given $f \in CC$ and $\mu \in CC'$, both $\mu * f$ and $f * \mu$ are continuous functions with absolutely and uniformly convergent Fourier series and their norms in C are bounded by $\|\mu\|_{CC'}\|f\|_{CC}$.*

PROOF.

(i) $$\begin{aligned}\|\tau_a f\|_{CC} &= \sum d(\gamma)\|\operatorname{tr}[(\tau_a f)^\wedge(\gamma)\gamma]\| \\ &= \sum d(\gamma)\|\operatorname{tr}[\hat{f}(\gamma)\tau_a\gamma]\| \\ &= \sum d(\gamma)\|\operatorname{tr}[\hat{f}(\gamma)\gamma]\| = \|f\|_{CC}\end{aligned}$$

and similarly for right translations.

To complete the proof of Part (i), define $\bar{\gamma}$ to be the member of Γ which is equivalent to the conjugate representation of γ as defined in [3, (27.27)]. Then $\gamma \in \Gamma$ if and only if $\bar{\gamma} \in \Gamma$ and moreover $\bar{\bar{\gamma}} = \gamma$. A simple calculation yields $(f^\vee)^\wedge(\gamma) = \bar{f}^\wedge(\gamma)^*$ and so

$$\operatorname{tr}[(f^\vee)^\wedge(\gamma)\gamma] = \overline{\operatorname{tr}[\bar{f}^\wedge(\gamma)\gamma^\vee]} = (\operatorname{tr}[\bar{f}^\wedge(\gamma)\gamma])^*.$$

Also $\mathrm{tr}[f^{\wedge}(\gamma)\gamma] = \overline{\mathrm{tr}[\hat{f}(\bar{\gamma})\bar{\gamma}]}$, and hence

$$\|f^{\vee}\|_{CC} = \|\check{f}\|_{CC} = \sum d(\gamma)\,\|\mathrm{tr}[\hat{f}(\bar{\gamma})\bar{\gamma}]\| = \|f\|_{CC}.$$

(ii) From the remarks preceding the statement of the lemma, $f * \mu$ has the formal Fourier series $\sum d(\gamma)\,\mathrm{tr}[\hat{\mu}(\gamma)\hat{f}(\gamma)\gamma(a)]$ at the point a. If we denote $d(\gamma)\mathrm{tr}[\hat{f}(\gamma)\gamma]$ by f_γ, then

$$\begin{aligned}
&(2.3) && |d(\gamma)\mathrm{tr}[\hat{\mu}(\gamma)\hat{f}(\gamma)\gamma(a)]| = |d(\gamma)\,\mathrm{tr}[\hat{\mu}(\gamma)\hat{f}_\gamma(\gamma)\gamma(a)]| \\
&(2.4) && = |d(\gamma)\,\mathrm{tr}[\hat{\mu}(\gamma)\,(\tau_a f_\gamma)^{\wedge}(\gamma)]| \\
&(2.5) && = |d(\gamma)\,\mathrm{tr}[\hat{\mu}(\gamma)\,(\tau_a f_\gamma)^{*\wedge}(\gamma)^{*}]| \\
&(2.6) && = |\langle(\tau_a f_\gamma)^{\vee}, \mu\rangle| \\
&(2.7) && \leqq \|(\tau_a f_\gamma)^{\vee}\|_{CC}\,\|\mu\|_{CC'} \\
&(2.8) && = \|f_\gamma\|_{CC}\,\|\mu\|_{CC'},
\end{aligned}$$

(2.6) following from (2.1) and (2.8) from (i) above. Thus the Fourier series of $f * \mu$ is absolutely and uniformly convergent and is bounded by $\sum_\gamma \|f_\gamma\|_{CC}\,\|\mu\|_{CC'} = \|f\|_{CC}\,\|\mu\|_{CC'}$. The corresponding result for $\mu * f$ is proved similarly.

THEOREM 2.3. *The following conditions on $E \subseteq \Gamma$ are equivalent.*

(i) *E is local Sidon.*

(ii) *Corresponding to each $\phi \in \mathfrak{E}^\infty$ there exists $\mu \in CC'$ such that $\phi = \hat{\mu}$ on E.*

(iii) *Corresponding to each $\phi \in \mathfrak{E}^\infty$ with $\phi(\gamma)$ unitary for each $\gamma \in E$, there exists $\mu \in CC'$ and $\delta > 0$ such that*

$$\|\phi(\gamma) - \hat{\mu}(\gamma)\|_{(\infty)} \leqq 1 - \delta \text{ for each } \gamma \in E.$$

PROOF. The proof that (i) implies (ii) carries over immediately from the proof of the corresponding result for Sidon sets (see [3, p. 417], for example). Clearly (ii) implies (iii) and we close the circle of implications by proving that (iii) implies (i). (This proof will also be similar to the proof for the Sidon case.)

Suppose that (iii) is valid and that $f \in CC_E$. Since f can always be written as $f_1 + if_2$ where each f_j satisfies $f_j = f_j^*$, in order to prove 2.1(ii) it suffices to show that $f \in A$ only in the case that $f = f^*$. Hence Theorem 2.1 will show that E is local Sidon. Corresponding to each $\gamma \in E$ there exists a unitary self-adjoint operator $W(\gamma)$ such that $W(\gamma)\hat{f}(\gamma) = |\hat{f}(\gamma)|$. From (iii) there exists $\mu \in CC'$ and $\delta > 0$ such that

$$\|W(\gamma) - \hat{\mu}(\gamma)\|_{(\infty)} \leqq 1 - \delta \text{ for each } \gamma \in E.$$

Let $\nu = \frac{1}{2}(\mu + \mu^*)$; then each $\hat{\nu}(\gamma)$ is self-adjoint and so

$$\| W(\gamma) - \hat{\nu}(\gamma) \|_{(\infty)} \leqq \tfrac{1}{2} \| W(\gamma) - \hat{\mu}(\gamma) \|_{(\infty)} + \tfrac{1}{2} \| W(\gamma)^* - \hat{\mu}(\gamma)^* \|_{(\infty)}$$
$$\leqq 1 - \delta \text{ for } \gamma \in E.$$

Put $g = f * \nu$; from Lemma 2.2(ii) we know that $g \in C$. Also $\mathrm{tr}[\hat{g}(\gamma)]$ is real-valued and moreover

$$\begin{aligned} |\mathrm{tr}[\hat{g}(\gamma)] - \mathrm{tr}[|\hat{f}(\gamma)|]| &= |\mathrm{tr}[\hat{\nu}(\gamma)\hat{f}(\gamma) - W(\gamma)\hat{f}(\gamma)]| \\ &\leqq \| \hat{\nu}(\gamma) - W(\gamma) \|_{(\infty)} \mathrm{tr}[|\hat{f}(\gamma)|] \\ &\leqq (1-\delta)\mathrm{tr}[|\hat{f}(\gamma)|], \end{aligned}$$

and so $\mathrm{tr}[\hat{g}(\gamma)] \geqq \delta\ \mathrm{tr}[|\hat{f}(\gamma)|]$ for $\gamma \in E$. Thus $\mathrm{tr}[\hat{g}(\gamma)]$ is real and non-negative for all $\gamma \in E$ and so from ([3, (34.9)] we have

$$\delta \sum d(\gamma)\mathrm{tr}[|\hat{f}(\gamma)|] \leqq \sum d(\gamma)\mathrm{tr}[\hat{g}(\gamma)] \leqq \| g \| < \infty.$$

Thus $f \in A$ and the proof is completed.

THEOREM 2.4. *Given $p \in (0, \infty)$, the following conditions on $E \subseteq \Gamma$ are equivalent.*

(i) *E is local $\Lambda(p)$.*

(ii) *There exists r in $(0, p)$ and $B = B(E)$ such that $\| f \|_{CL^p} \leqq B \| f \|_{CL^r}$ for all f in CL^r.*

(iii) *For every r in $(0, p)$ there exists $B = B(E, r)$ such that $\| f \|_{CL^p} \leqq B \| f \|_{CL^r}$ for all f in CL^r.*

(iv) *$CL^r = CL^p$ for all $r \in (0, p)$.*

(v) *There exists r in $(0, p)$ and $B = B(E)$ such that $\| f \|_p \leqq B \| f \|_{CL^r}$.*

(vi) *For every r in $(0, p)$ there exists $B = B(E, r)$ such that $\| f \|_p \leqq B \| f \|_{CL^r}$.*

Apart from a use of Proposition 1.1(i), the proofs are similar to those of Theorem 2.1 and hence will be omitted.

3. Examples and applications

Sufficient conditions. Let $E \subseteq \Gamma$; then $f \in d(\gamma)\mathrm{tr}[\hat{f}(\gamma)\gamma]$, $\gamma \in E$, satisfies

$$\begin{aligned} \| f \|_A &= d(\gamma)\mathrm{tr}[|\hat{f}(\gamma)|] \\ &\leqq d(\gamma) \| f \|_2 \text{ by [3, (D.51)]}, \\ &\leqq d(\gamma) \| f \|. \end{aligned}$$

Thus E is local Sidon and hence local $\Lambda(p)$ whenever

$$\sup\{d(\gamma) \colon \gamma \in E\} \text{ is finite.} \tag{3.1}$$

3.2 *Necessary conditions.* An example due to Figà-Talamanca and Rider [2] shows that, in general, condition (3.1) is not necessary for a set to be local Sidon (or even Sidon). However Cecchini [1] has shown that when B is a compact Lie group, then (3.1) is a necessary condition for a set to be local $\Lambda(4)$, and hence for a set to be local Sidon. Thus we have proved the statement in the abstract since whenever G is a compact Lie group with $E = \Gamma$ satisfying (3.1), then Γ is not local Sidon and so from Theorem 2.1 there exists f in CC (and hence the Fourier series of f is uniformly and absolutely convergent) such that f is not a member of A. Also the method of Figà-Talamanca as described in [3, (37.23)], for example, shows that the duals of certain infinite products of finite groups contain sets which are not local Sidon or local $\Lambda(p)$, $p > 0$.

The author was not able to decide the validity of the statement that Γ is never local Sidon (or local $\Lambda(p)$) whenever the degrees of the representations in Γ are unbounded.

Added in Proof

M. A. Picardello (*Some random Fourier series on compact noncommutative groups*, to appear) has also considered this question and has given a sufficient condition for a set to be local $\Lambda(p)$ in terms of Steinhaus random Fourier series. He has also shown that the dual of an infinite product of non-abelian compact groups is not local $\Lambda(4)$.

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